Chapter 5

Counting

There are three types of people in this world: Those who can count, and those who can't.

Counting seem quite simple but this is quite deceptive, especially when we have complicated system. If you do not believe me have a look at the probability section. To make like a little simpler we lay down some rules.

Sets

If we have two sets A and B the number of item in the sets (the cardinality) is written $\|A\|$ and $\|B\|$. Then we can show that

$$\|A \cup B\| = \|A\| + \|B\| - \|A \cap B\|$$

This is fairly easy to see if you use a Venn diagram. For 3 sets

$$\|A \cup B\| = \|A\| + \|B\| + \|C\| - \|A \cap B\| - \|B \cap C\| - \|A \cap C\| + \|A \cap B \cap C\|$$

Example

Let S be the set of all outcomes when two dice (one blue ; one green) are thrown. Let A be the subset of outcomes in which both dice are odd, and let B be the subset of outcomes in which both dice are even. We write C for the set of outcomes when the two dice have the same number showing.

How many elements are there in the following sets?

It is useful to have the set S set out as below

1 1	1 2	1 3	1 4	1 5	1 6
2 1	2 2	2 3	2 4	2 5	2 6
3 1	3 2	3 3	3 4	3 5	3 6
4 1	4 2	4 3	4 4	4 5	4 6
5 1	5 2	5 3	5 4	5 5	5 6
6 1	6 2	6 3	6 4	6 5	6 6

then we have

1. $\|A\| = 9$
2. $\|B\| = 9$
3. $\|C\| = 6$
4. $\|A \cap B\|$=0
5. $\|A \cup B\| = 18$
6. $\|A \cap C\| = \|(1, 1), (3, 3), (5, 5)\| = 3$
7. $\|A \cup C\| = \|A\| + \|C\| - \|A \cap C\| = 9 + 6 - 3 = 12$

Chains of actions

If we have to perform two actions in sequence and the first can be done m ways while the second can be done in n there will be mn possibilities in total.

- Suppose we wish to pick 2 people from 9. The first can be picked in 9 ways the second in 8 giving $9 \times 8 = 72$ possibilities in total.
- If we roll a die and then toss a coin there are $6 \times 2 = 12$ possibilities.

This extends to several successive actions. Thus

1. If we roll a die 3 times then there are $6 \times 6 = 216$ possibilities.

2. If we toss a coin 7 times there are $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7 = 128$ possibilities.

3. My bicycle lock has 4 rotors each with 10 digits. That gives $10 \times 10 \times 10 \times 10 = 10^4$ combinations.

4. Suppose you have to provide an 8 character password for a credit card company. They say that you can use a to z (case is ignored) and 0 to 1 but there must be at least one number and at least one letter.

   there are 26 letters and 10 numbers so you can make 8^{36} possible passwords. Of these there are 8^{10} which are all numbers and 8^{26} which are all letters. This gives $8^{36} - 8^{26} - 8^{10} = 3.245 \times 10^{32}$ allowable passwords.

Permutations

Suppose I have n distinct items and I want to arrange them in a line. I can do this in

$$n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 3 \times 2 \times 1$$

We compute this product so often it has a special symbol $n!$. However to avoid problems we *define*

$$1! = 0 \text{ and } 0! = 1$$

So $3! = 3 \times 2 \times 1 = 6$ while $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

If we look at the characters in (1D4Y) there are $4! = 24$ possible distinct arrangements.

Sometimes we do not have all distinct items. We might have n item of which r are identical then there are $n!/r!$ different possible arrangements. So WALLY can be arranged in $5!/2! = 60$ ways.

It is simpler to just state a rule in the more general case:

Suppose we have n objects and

- there are n_1 of type 1.
- there are n_2 of type 2.
- $\cdots\cdots$
- there are n_k of type k.

The total number of items in n, so $n = n_1 + n_2 + \cdots n_k$ then there are

$$\frac{n!}{n_1!n_2!n_3!\cdots n_k!}$$

possible arrangements.

Suppose we have 3 white, 4 red and 4 black balls. They can be arranged in a row in

$$\frac{11!}{3!4!4!} = 11550$$

possible ways while the letters in WALLY can be arranged in

$$\frac{5!}{2!1!1!1!} = 60 \text{ ways}$$

Combinations

The number of ways of picking k items from a group of size n is written $\binom{n}{k}$ or (for the traditionalists) ${}_nC_k$. The definition is

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

So the number of ways of picking 5 students from a group of 19 is

$$\binom{19}{5} = \frac{19!}{5!14!} = \frac{19 \times 18 \times 17 \times 16}{4 \times 3 \times 2 \times 2}$$

Examples

1. Suppose you want to win the lottery. There are 49 numbers and you can pick 6. This can be done in

$$\frac{49!}{6!43!} = 13983816 \text{ ways}$$

so your chances of a win are 1/13983816.

2. How many ways can you pick 5 correct numbers in the lottery. There are $\binom{6}{5}$ ways to pick the 5 correct numbers and 49-6=43 ways of picking the remaining number. This gives 6×43 ways.

3. When we pick 3 correct numbers there are $\binom{6}{3}$ ways of picking the winning numbers and $\binom{43}{3}$ ways of picking the losing ones. This gives $\binom{6}{3} \times \binom{43}{3} = 20 \times 12341 = 246820$ ways in all.

5.0.7 Binomial Expansions

Now we have combinations we can examine a very useful result known as the binomial expansion. To start we can show that

$$(a+b)^2 = a^2 + 2ab + b^2$$

and

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

In general we can prove that for an integer $n > 0$

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}ab^{n-1} + b^n$$

or

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i}b^i.$$

This can be done by induction, but there isis a page or so of algebra!

For example

$$(2+x)^5 = 2^5 + \binom{5}{1}2^4x + \binom{5}{2}2^3x^2 + \binom{5}{3}2^2x^2 + \binom{5}{4}2x^4 + x^5$$

or

$$(2+x)^5 = 2^5 + 5 \times 2^4x + 10 \times 2^3x^2 + 10 \times 2^2x^2 + 5 \times 2x^4 + x^5$$

Suppose you were given $\left(3x+5/x^3\right)^8$ and you wanted the term in the expansion which *did not* have an x. From the above the general term is

$$\binom{8}{i}(3x^3)^{8-i}(5/x^3)^i.$$

The x terms cancel when $8-i=3i$ or $i=2$. Then the term is

$$\binom{8}{2}(3x^3)^6(5/x^3)^2 = \binom{8}{2}3^6 5^2$$

We can do something similar for non-integral n as follows:

$$(1+x)^n = 1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}+\cdots+\frac{n(n-1)(n-2)\cdots(n-k+1)}{1.2.3\cdots k}x^k+\cdots$$

but this is only true when $|x|<1$.

Thus $(1+x)^{1/2} = 1+\left(\frac{1}{2}\right)x^{1/2}+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^{-1/2}+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-3/2}+\ldots$

Examples

1. Suppose we look at sports scholarships awarded by American universities. A total of 147,000 scholarships were earned in 2001. Out of the 5,500 scholarships for athletics, 1500 were earned by women. Women earned 75,000 scholarships in total. How many men earned scholarships in athletics?

2. In clinical trials of the suntan lotion, Delta Sun, 100 test subjects experienced third degree burns or nausea (or both). Of these, a total of 35 people experienced third degree burns, and 25 experienced both third degree burns and nausea. How many subjects experienced nausea?

3. A total of 1055 0 MSc degrees were earned in 2002. Out of the 41 MSc degrees in music and music therapy, 5 were earned by men. Men earned 650 MSc degrees. How many women earned MSc degrees in fields other than music and music therapy?

4. A survey of 200 credit card customers revealed that 98 of them have a Visa account, 113 of them have a Master Card, 62 of them have a Visa account and a American Express, 36 of them have a Master Card account and an American Express, 47 of them have only a Master Card account, 32 have a Visa account and a Master Card account and an American Express. Assume that every customer has at least one of the services. The number of customers who have only have a Visa card is?

5. So for example from the New York Times According to a New York Times report on the 16 top-performing restaurant chains

   (a) 11 serve breakfast.
   (b) 11 serve beer.
   (c) 10 have full table service i.e. they server alcohol and all meals.

   All 16 offered at least one of these services. A total of 5 were classified as "family chains," meaning that they serve breakfast, but do not serve alcohol. Further a total of five serve breakfast and have full table service, while none serve breakfast, beer, and also have full table service. We ask

   (a) (How many serve beer and breakfast?
   (b) How many serve beer but not breakfast?
   (c) How many serve breakfast, but neither have full table service, nor serve beer?
   (d) How many serve beer and have full table service?

6. When $\mid x \mid < 1$ then show that

   - $1/(1-x) = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$
   - $1/(1-x)^{1/2} = 1 + (1/2)x + \frac{(1/2)(-1/2)x^2}{1.2} + \frac{(1/2)(-1/2)(-3/2)x^3}{1.2.3} + x^4 + \cdots$
   - $1/(1-x)^2 = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots + nx^{n-1} + \ldots$

7. Expand $(1+2x)^7$

8. Which is the coefficient of the term without an x in $(x + 2/x)^{11}$.

9. Find an approximation for $(0.95)^{11}$.

10. Find the first 3 terms of the expansion of $(1+x)^{1/4}$.